Computation of Fields in an Arbitrarily Shaped Heterogeneous Dielectric or Biological Body by an Iterative Conjugate Gradient Method

JOHNSON J. H. WANG, SENIOR MEMBER, IEEE, AND JOHN R. DUBBERLEY

Abstract —**Electromagnetic (EM) fields in a three-dimensional, arbitrarily shaped heterogeneous dielectric or biological body illuminated by a plane wave are computed by an iterative conjugate gradient method. The method is a generalized method of moments applied to the volume integral equation. Because no matrix is explicitly involved or stored, the present iterative method is capable of computing EM fields in objects an order of magnitude larger than those that can be handled by the conventional method of moments. Excellent numerical convergence is achieved. Perfect convergence to the result of the conventional moment method using the same basis and weighted with delta functions is consistently achieved in all the cases computed, indicating that these two algorithms (direct and iterative) are equivalent.**

I. Introduction

THE COMPUTATION of electromagnetic fields in a three-dimensional, arbitrarily shaped dielectric or biological body at frequencies in the resonance region or lower has been carried out primarily by the conventional method of moments [1]–[6]. Other numerical methods also exist but they have been essentially limited to two-dimensional problems except for some recent finite element and finite difference techniques [7], which have been applied to frequencies far below the resonance region.

A major limiting factor of the conventional method of moments [8] is its need for a large computer memory to store the matrix involved in the computation. For a large mainframe computer, such as the CDC Cyber 855, a matrix of about 240×241 represents a practical upper limitation. The use of virtual memory or a sparse matrix technique allows for larger matrices, but the execution time increases at an astronomical rate, often rendering these techniques impractical [9]. Recently, the possibility of using iterative methods for large bodies was investigated because they can be carried out without the direct involvement of a large matrix, which rapidly exhausts the computer's memory. As a result, the iterative methods can solve problems involving a larger number of unknowns than the conventional method of moments by at least an order of magnitude [10]–[13]. However, the iterative method has only been successfully applied to two-dimensional problems [12], [13]. Computations for three-dimensional problems by iterative methods incur difficulties in achieving convergence [14].

In this paper, we discuss an iterative computer algorithm and its numerical results for the problem of a three-dimensional, arbitrarily shaped dielectric or biological body under plane wave illumination. Good numerical convergence and excellent agreement with the conventional method of moments were achieved.

Manuscript received September 23, 1988; revised February 6, 1989.

The authors are with the Georgia Tech Research Institute, Georgia Institute of Technology, Atlanta, GA 30332.

IEEE Log Number 8927785.

II. Formulation of the Problem

Fig. 1 shows a three-dimensional, arbitrarily shaped heterogeneous dielectric or biological body illuminated by a plane wave $\boldsymbol{E}^i$. The problem can be formulated by replacing the material body occupying V by an equivalent volume current $\boldsymbol{J}$ as shown. A volume integral equation can be written as follows [1]:

$$\int_v \boldsymbol{J}(\boldsymbol{r}')\cdot\underline{\boldsymbol{G}}_e(\boldsymbol{r},\boldsymbol{r}')\,dv' + D(\boldsymbol{r})\boldsymbol{J}(\boldsymbol{r}) = -\boldsymbol{E}^i(\boldsymbol{r}) \qquad (1)$$

where

$$\underline{\boldsymbol{G}}_e = -j\omega\mu_0\left(\underline{\boldsymbol{I}} + \frac{1}{k^2}\nabla\nabla\right)g(\boldsymbol{r},\boldsymbol{r}') \qquad (2)$$

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \qquad (3)$$

$$\underline{\boldsymbol{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \qquad (4)$$

$$D(\boldsymbol{r}) = -\frac{\epsilon_r(\boldsymbol{r})+2}{3j\omega[\epsilon(\boldsymbol{r})-\epsilon_0]}. \qquad (5)$$

Here ϵ is the permittivity of the medium, ϵ_r is the relative permittivity, and $\epsilon=\epsilon_r\epsilon_0$. $\boldsymbol{r}$ and $\boldsymbol{r}'$ denote the position vectors at the field and source points respectively. $k^2=\omega^2\epsilon_0\mu_0$ and the hat "$\hat{}$" denotes a unit vector. The symbol $\int$ denotes a principal-value integration with an infinitesimal sphere centered at $\boldsymbol{r}=\boldsymbol{r}'$ extracted. The issues regarding the handling of the singularity in this case have been discussed in [1], followed by a number of papers in the literature (e.g., [15]).

0018-9480/89/0700-1119$01.00

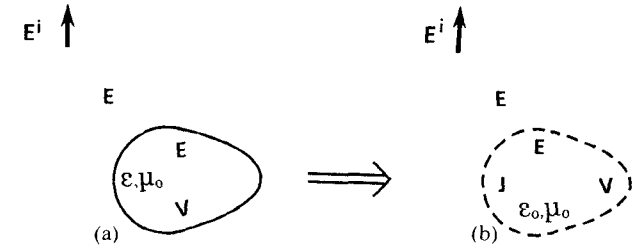

Fig. 1. Replacement of an inhomogeneous dielectric of biological body by an equivalent volume current.

Equation (1) can be written in the following form:

$$\int_v \boldsymbol{J}(\boldsymbol{r}')\cdot\underline{\boldsymbol{K}}(\boldsymbol{r},\boldsymbol{r}')\,dv' = -\boldsymbol{E}^i(\boldsymbol{r}) \tag{6}$$

which is a more convenient form for later discussion of the iterative method.

III. Discretization by Volume Cells

Solution of the integral equation, in the form of either (1) or (6), begins with the process of discretizing the volume V into L cubic volume cells $V_1, V_2, \cdots, V_L$, generally of different cell sizes, as reported in [1], [4], and [16]. The discretization is carried out by expressing the equivalent volume current $\boldsymbol{J}$ as

$$\boldsymbol{J}(\boldsymbol{r}) = \sum_{l=1}^{L}\sum_{k=1}^{3} J_l^k \boldsymbol{B}_l^k(\boldsymbol{r}) \tag{7}$$

where

$$\boldsymbol{B}_l^k(\boldsymbol{r}) = \hat{u}_k B_l^k(\boldsymbol{r}) = \hat{u}_k P_m(\boldsymbol{r}) \tag{8}$$

$$P_m(\boldsymbol{r}) = \begin{cases} 1 & \boldsymbol{r}\in V_m \\ 0 & \text{elsewhere.} \end{cases} \tag{9}$$

Here $\hat{u}_k$ are unit vectors, which are $\hat{x}$, $\hat{y}$, and $\hat{z}$ for $k=1$, 2, and 3, respectively, in rectangular coordinates.

In [1], [4], and [16] the conventional method of moments [8] was employed with $\boldsymbol{B}_l^k$ as the basis functions. The weighting functions are chosen as

$$\boldsymbol{W}_m^k(\boldsymbol{r}) = \delta(|\boldsymbol{r}-\boldsymbol{r}_m|)\,\hat{u}_k. \tag{10}$$

By performing a symmetric product with $\boldsymbol{W}_m^k$ on (1) with $\boldsymbol{J}$ discretized by (7), for $m=1,\cdots,L$ and $k=1,2,3$, one obtains $3L$ linear equations, or a $3L\times(3L+1)$ matrix equation, which can be solved for the unknown J_l^k.

IV. Iterative Solution by the Conjugate Gradient Method

As previously noted, the computer memory requirement due to the $3L\times(3L+1)$ matrix in the conventional method of moments is a serious limiting factor. The use of iterative methods can circumvent this difficulty and allow the number of unknowns to increase by at least an order of magnitude. While other iterative methods often suffer from uncertainties in achieving numerical convergence, the conjugate gradient method has been noted for its assured convergence in at most N steps, where N is the number of basis functions or unknowns.

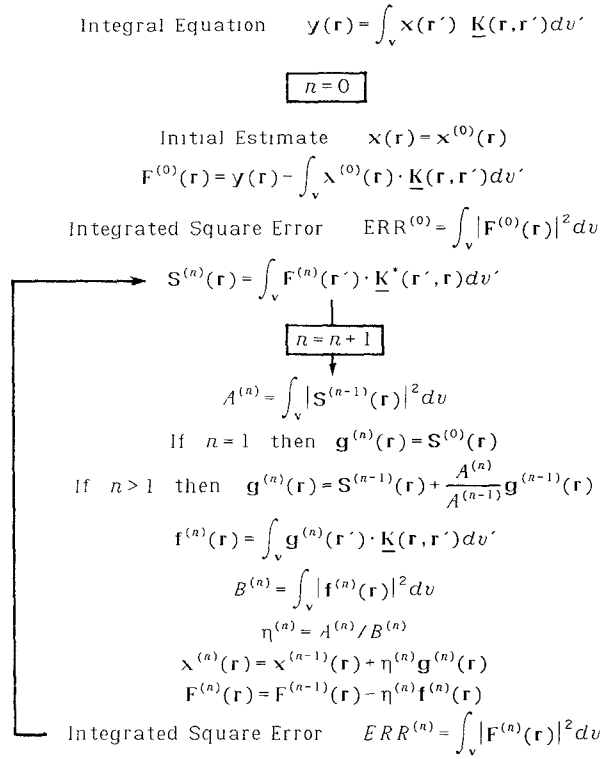

Fig. 2. An iterative conjugate gradient algorithm for an integral equation pertaining to a three-dimensional problem.

Even though the application of the conjugate gradient method has been successful in one- and two-dimensional problems, its use in three-dimensional problems has experienced difficulties. First, there appears to be unexplainable difficulties in achieving numerical convergence [14]. Second, when the number of basis functions, N, is large, the computation is very costly if N iterations are needed for convergence. (Fortunately, numerical convergence in one- and two-dimensional problems generally arrives after about $N/6$ iterations [14].) Numerical experiments with our newly developed conjugate gradient algorithm have shown that rapid convergence can be consistently achieved in about $N/6$ iterations.

The iterative algorithm is an extension of van den Berg's two-dimensional conjugate gradient scheme [13] to three-dimensional problems. Fig. 2 shows the basic scheme which minimizes the integrated square error in the iterative process. The normalized integrated square error is defined as

$$\text{ERRN} = \frac{\text{ERR}^{(N)}}{\int_v |\boldsymbol{x}(\boldsymbol{r}')|^2\,dv'}. \tag{11}$$

An immediate question is whether this iterative method would lead to results different from the conventional method of moments using the same basis functions as discussed in Section III. Our answer is that in general the results between direct and iterative methods are different because a different choice of weighting functions would

TABLE I
COMPARISON OF RESULTS BETWEEN DIRECT AND ITERATIVE MM COMPUTATIONS

CELL NO	DIRECT MM		ITERATIVE MM		
			1ST ITERATION	2ND ITERATION	3RD ITERATION
			NUMBER OF INTERATION =1 MEAN SQ. ERROR = .1526E+02	NUMBER OF INTERATION =2 MEAN SQ. ERROR = .5937E+01	NUMBER OF INTERATION =3 MEAN SQ. ERROR = .5528E-21
1	.23947E+00	.24262E+00	-.77133E+00 .76836E-02	-.62023E+00 .15577E-01	.23947E+00 .24262E+00
2	.40630E+00	.40664E+00	-.11032E+01 -.16759E-02	-.98807E+00 .34514E-02	.40630E+00 .40664E+00
3	.49628E+00	.49401E+00	-.10214E+01 -.28334E-02	-.11734E+01 -.85615E-02	.49628E+00 .49401E+00
4	.49628E+00	.49401E+00	-.10214E+01 -.28334E-02	-.11734E+01 -.85615E-02	.49628E+00 .49401E+00
5	.40630E+00	.40664E+00	-.11032E+01 -.16759E-02	-.98807E+00 .34514E-02	.40630E+00 .40664E+00
6	.23947E+00	.24262E+00	-.77133E+00 .76836E-02	-.62023E+00 .15577E-01	.23947E+00 .24262E+00

lead to a different matrix and thus different results. However, in the present case, the use of pulse functions as the basis and delta functions for weighting leads to the same numerical results as the iterative conjugate gradient method using the same basis functions.

The close relationship between the conventional method of moments and the iterative method is discussed in a separate paper by the first author, who finds it useful, practical, and appropriate to include these two methods, as well as the reaction integral equation method, within the broad context of the "generalized method of moments" [17].

At this point, it may be worthwhile to emphasize that some of the iterative methods in the literature are directly and intimately associated with a matrix representation of the operator equation, while in the present algorithm no matrix is involved. Of course, one can always relate the present iterative approach to a matrix, but such a connection is unnecessary and is only useful as a tool in examining certain properties of the method. The direct dependence on the matrix formulation in some other iterative methods is the reason why they do not have the advantage of reduced memory requirements in their computational process.

V. Numerical Results and Observations

A general computer program based on the techniques just described was written and used to solve the problem of field intensity for a number of dielectric or biological bodies. Based on our computational results, we observed that numerical convergence is always achieved in about N iterations, as it should be, where N is the number of unknowns or basis functions.

Table I shows a comparison of numerical results from the conventional MM and the present iterative MM for a dielectric cylinder with a relative permittivity of $71.7 - j6.53$. The cylinder is 43.4 mm long and has a square cross section with 7.24 mm on each side. It is illuminated by a plane wave of unit amplitude at 2450 MHz with its polarization parallel to, and direction of propagation perpendic-

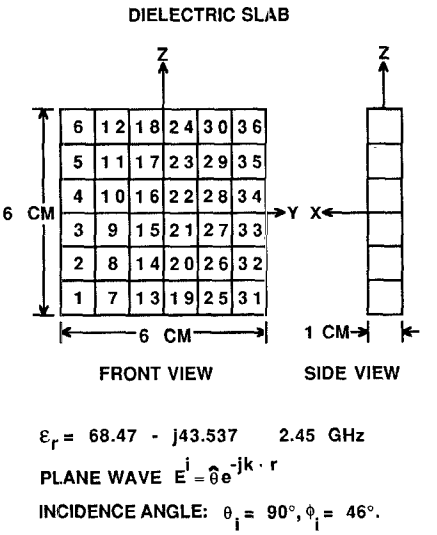

Fig. 3. A square slab illuminated by a plane wave is divided into 36 cubic cells with cell numbers shown.

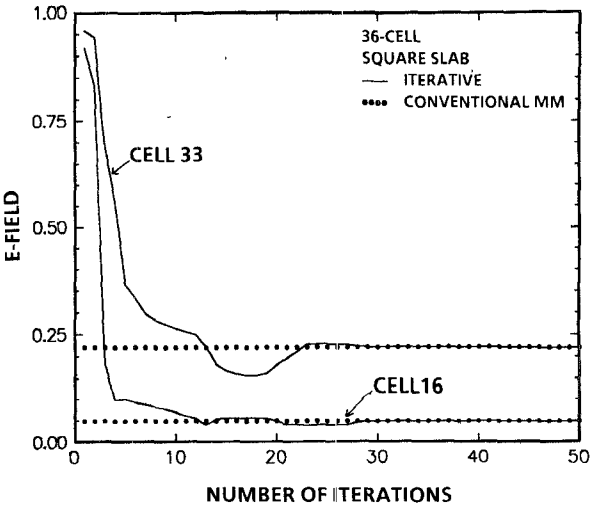

Fig. 4. Numerical convergence of electric fields for the square slab of Fig. 3.

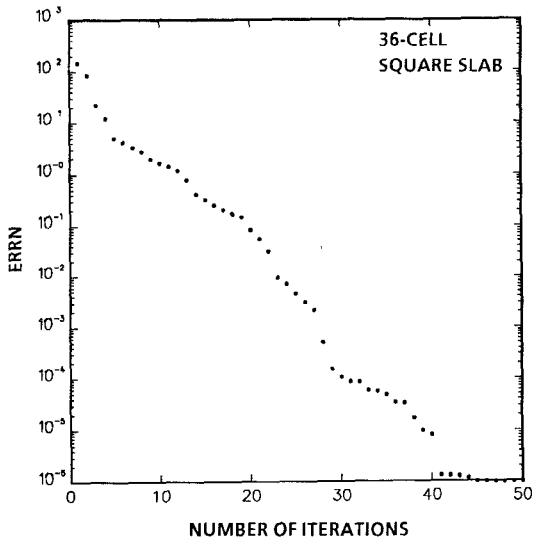

Fig. 5. Numerical convergence for the square slab of Fig. 3.

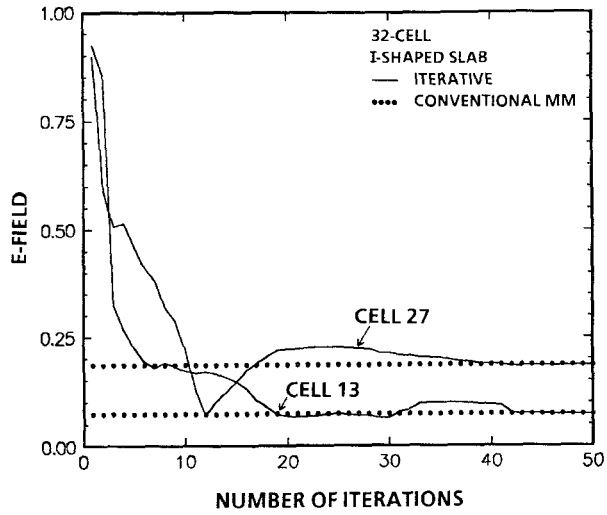

Fig. 7. Numerical convergence of electric fields for the I-shaped slab of Fig. 6.

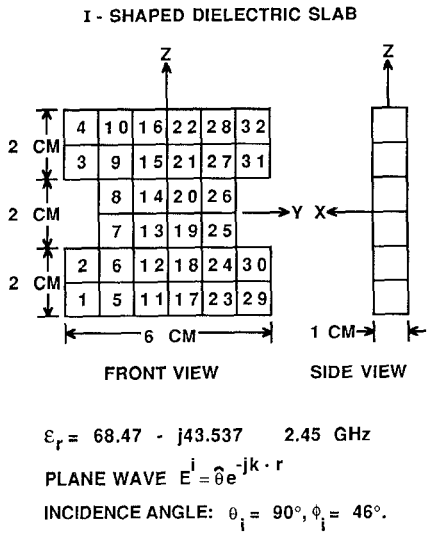

Fig. 6. An I-shaped dielectric slab illuminated by a plane wave is divided into 32 cubic cells with cell numbers shown.

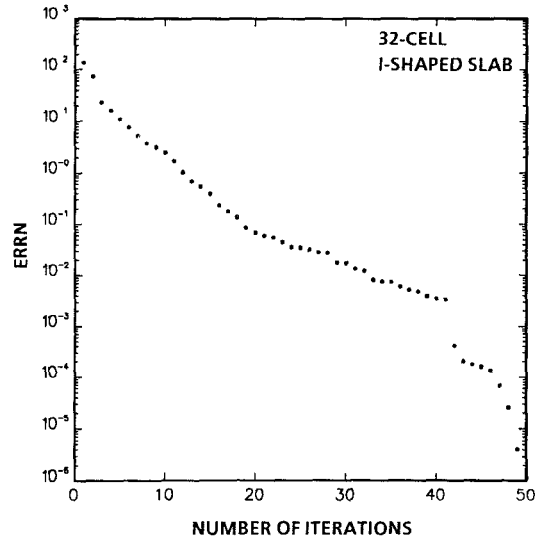

Fig. 8. Numerical convergence for the I-shaped slab of Fig. 6.

ular to, the cylinder axis. The cylinder is discretized into six cubic cells, 1 through 6, with cell 1 and cell 6 being the end cells. The computed complex magnitude of the component of the election field parallel to the cylinder axis in cells 1 through 6 is displayed in Table I. As can be seen, excellent convergence is achieved after only three iterations in the iterative MM, with the mean square error dropping from 5.937 after the second iteration to 0.5528×10^{-21} after the next (third) iteration. It is also significant that the difference in final results between the direct and iterative MM is so small that it is beyond the five significant numbers displayed in the table. In this particular case, the equivalence between the direct and iterative MM is clearly demonstrated.

Fig. 3 shows a square dielectric slab illuminated by a plane wave, which is 1 V/m in electric field intensity in all the computations in this paper. The $\boldsymbol{E}$ field is θ-polarized and propagates in the direction of $\theta_i = 90°$ and $\phi_i = 46°$, which corresponds to a direction of rapid changes in the scattered field. The slab is divided into 36 cells, each of which is numbered for reference.

On a CDC Cyber 855, the present computer program can handle a total of about 11 000 unknowns, or 3666 cells. As the number of unknowns increases, the computer execution time increases rapidly. It is usually impractical and expensive in the case of large scatterers to carry out more iterations than necessary, even though additional iterations help to establish evidence of convergence. The general consensus in one- and two-dimensional problems is that $N/6$ iterations are usually adequate, where N is the number of unknowns or basis functions [14]. We have also observed in our computation of three-dimensional problems that $N/6$ iterations are usually sufficient and that the

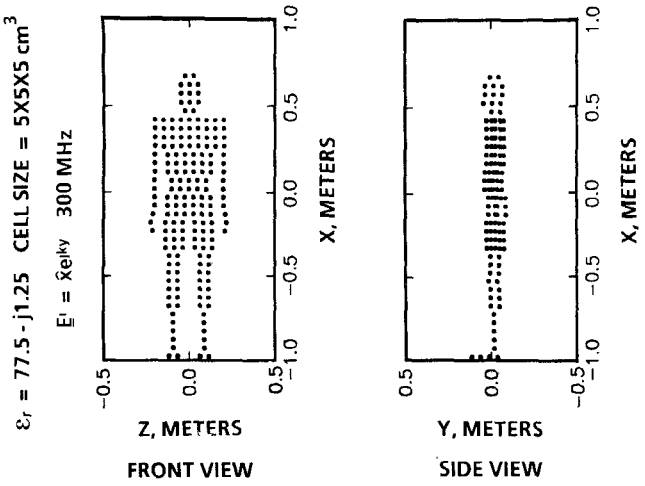

Fig. 9. Front and side views of a 423-cell human body illuminated by a plane wave.

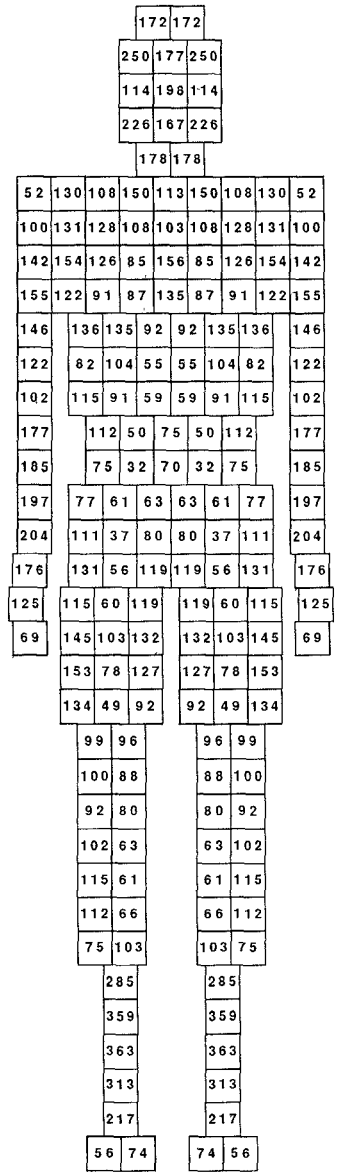

Fig. 10. Electric field intensity (in mV/m) in each cell on the $y = 0$ plane of the human body of Fig. 9.

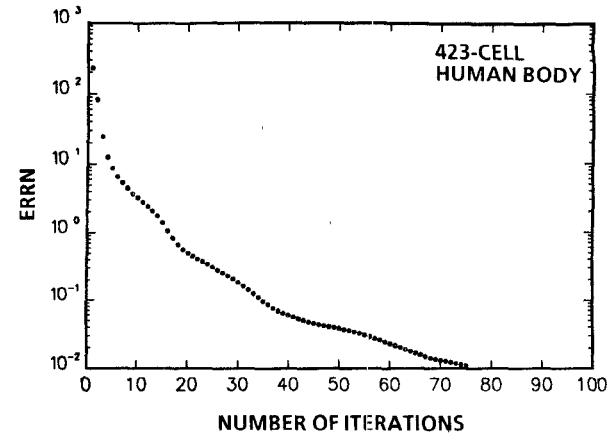

Fig. 11. Numerical convergence for the human-body problem of Fig. 9

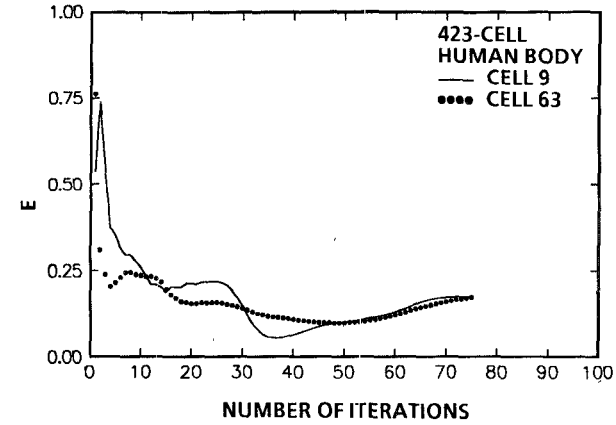

Fig. 12. Numerical convergence of the electric fields for the human body of Fig. 9.

rapidity of convergence depends on the geometry of the scatterer and the polarization and angle of incidence of the illuminating field.

Fig. 4 shows the computed electric fields in cell 16 and cell 33 of the dielectric slab in Fig. 3 as a function of the number of iterations. As can be seen, good results are obtained after 22 iterations, slightly more than the $N/6$, or 18, iterations that are usually needed. Fig. 5 shows the normalized integrated square error (ERRN) of the electric fields versus the number of iterations. As can be seen, the normalized integrated square error becomes very small long before N or 98 iterations are carried out.

Fig. 6 shows a 32-cell I-shaped dielectric slab illuminated by a plane wave propagating in the direction $\theta_i = 90°$ and $\phi_i = 46°$. Each cubic cell is 1 cm on each side and is numerically numbered as shown. The convergence of electric field computation for this case is exhibited in Fig. 7 for cell 13 and cell 27. As can be seen, the convergence is slower than that of the square slab, because the geometry of the I-shaped slab is more complex. The ERRN versus the number of iterations for this case is shown in Fig. 8.

Although the present algorithm can handle up to 3666 cells on a CDC Cyber 855, as compared to about 80 cells or fewer for the conventional moment method algorithm,

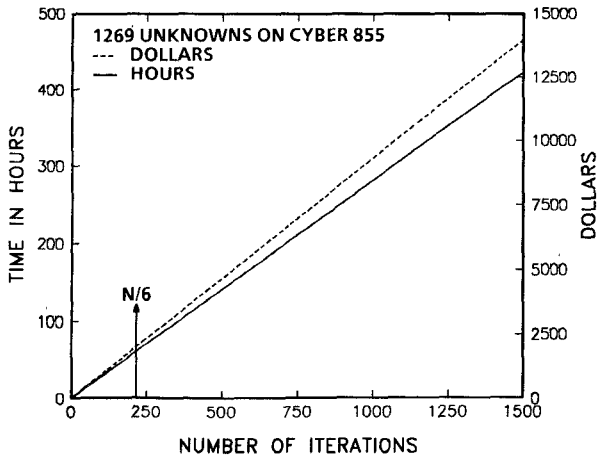

Fig. 13. An estimate for the execution time and dollar cost.

we have not tried to test the maximum capacity of the computer because of the high cost involved. To demonstrate that the present algorithm can deal with scatterers much larger than the conventional moment method, we computed the electric fields inside a human body with its front and side views shown in Fig. 9. The human body is divided into 423 cells with 1269 unknowns. The dots are the centers of the 423 cubic cells which make up the human body. The plane wave is x-polarized and propagates in the y direction.

The computed electric field intensity (in mV/m) for each cell on the $y = 0$ plane of the human body of Fig. 9 is exhibited in Fig. 10. A total of 75 iterations was predetermined in the computation to limit the cost of the computer run. In practice, the accuracy in energy deposition in biological bodies is not critical and errors of 10 to 20 percent are usually acceptable. As shown in Fig. 11 the ERRN is reduced to an acceptable level of 0.01126 after 75 iterations. The convergence of the electric field intensity is exemplified in Fig. 12 for cell 9 and cell 63, which are near the center of the head and the center of the heart, respectively.

Since the computer execution time is a consideration of practical importance, we have made an estimated run time and dollar cost on the CDC Cyber 855 at Georgia Tech for the 423-cell problem, as shown in Fig. 13. It provides a basis for simple and direct cost estimates for those who are concerned with the costs in computing for large bodies. As can be seen, the cost can become prohibitive if the number of unknowns is large. Fortunately, usually $N/6$ iterations are sufficient. In addition, there are techniques to reduce the cost and run time, which we plan to implement in the near future.

VI. Conclusions

An iterative conjugate gradient algorithm has been developed to compute the fields in a three-dimensional, arbitrarily shaped heterogeneous dielectric or biological body. Excellent numerical convergence behavior is observed in our computations.

The present algorithm can handle unknowns which are an order of magnitude larger in number than those that can be computed by the conventional moment method. A case involving a 423-cell human body is computed to demonstrate this feature.

References

[1] D. E. Livesay and K. M. Chen, "Electromagnetic fields induced inside arbitrarily shaped biological bodies," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp 1273–1280, Dec. 1974.

[2] B. J. Guru and K. M. Chen, "Experimental and theoretical studies on electromagnetic fields induced inside biological bodies," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-24, pp. 433–440, July 1976.

[3] K. M. Chen and B. S. Guru, "Internal EM field and absorbed power density in human torsos induced by 1–500-MHz EM waves," *IEEE Trans. Microwave Tech.*, vol. MTT-25, pp. 746–756, Sept. 1977.

[4] G. W. Hohmann, "Three-dimensional induced polarization and electromagnetic modeling," *Geophysics*, vol. 40, pp. 309–324, Apr 1975.

[5] H. Massoudi, C. H. Durney and M. F Iskander, "Limitations of the cubical block model of man in calculating SAR distributions," *IEEE Trans. Microwave Theory Tech.*, vol MTT-32, pp. 746–756, Aug. 1984.

[6] M. J. Hagmann, O. P. Gandhi, and C. H. Durney, "Numerical calculations of electromagnetic energy deposition for a realistic model of man," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-27, pp. 804–809, Sept 1979.

[7] K D. Paulsen, D. R. Lynch, and J. W. Strohbehn, "Three-dimensional finite, boundary and hybrid element solutions of the Maxwell equations for lossy dielectric media," *IEEE Trans Microwave Theory Tech.*, vol. 36, pp. 682–693, Apr. 1988.

[8] R. R. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.

[9] J. J. H. Wang and C. Papamcolopulos, "A study of the analysis and measurements of three-dimensional arbitrarily-shaped dielectric scatterers," Final Report RADC-TR-80-372, Rome Air Development Center, Griffiss AFB, NY, Dec 1980. (Partially presented at the 1982 URSI Meeting, Albuquerque, NM.)

[10] A. F. Peterson and R. Mittra, "Method of conjugate gradients for the numerical solution of large-body electromagnetic scattering problems," *J. Opt Soc Amer.*, vol. 2, pp. 971–977, June 1985.

[11] T. K. Sarkar, "Application of the fast Fourier transform and the conjugate gradient method for efficient solution of electromagnetic scattering from both electrically large and small conducting bodies," *Electromagnetics*, vol. 5, pp. 99–122, 1985.

[12] P. M. van den Berg, "Iterative computational techniques in scattering based upon the integrated square error criterion," *IEEE Trans. Antennas Propagat.* vol. AP-32, pp 1063–1071, Oct. 1984.

[13] P. M. van den Berg, A. T de Hoop, A Segal, and N. Praagman, "A computational model of the electromagnetic heating of biological tissue with application to hyperthermia cancer therapy," *IEEE Trans. Biomed Eng.*, vol BME-30, pp. 797–805, Dec. 1983.

[14] Panel Discussions in Session 16, "Special session on iterative methods," 1988 joint IEEE AP-S Symposium and National Radio Science Meeting, Syracuse, NY, June 7, 1988.

[15] J. J. H. Wang, "A unified and consistent view on the singularities of the electric dyadic Green's function in the source region," *IEEE Trans. Antennas Propagat.*, vol. AP-30, pp 463–468, May 1982.

[16] J. J. H. Wang, F. L. Cain, and E. C Burdette, "Numerical modeling of three-dimensional arbitrarily-shaped heterogeneous biological bodies under complex excitations," presented at the 1978 Int. Symp. Biological Effects of EM Waves, Ottawa, Canada, June 1978.

[17] J. J. H. Wang, "Generalization and interpretation of the moment method in electromagnetics," submitted to *IEEE Trans. Microwave Theory Tech.*

Johnson J. H. Wang (M'68–SM'79) was born in Hunan, China, on Oct. 24, 1938. He received the B.S.E.E. degree from National Taiwan University, the M.S. degree from Florida State University, Tallahassee, FL and the Ph.D. degree from the Ohio State University, Columbus, in 1962, 1965, and 1968, respectively.

From 1968 to 1975, he worked at a number of industrial firms. In 1975, he joined the Georgia Institute of Technology, Atlanta, GA. His research interests include antennas, bioelectromagnetics, radiation and scattering measurements, microwave imaging, and numerical methods in solving electromagnetic problems.

John R. Dubberley was born in Opelika, AL, on May 28, 1966. He received the B.E.E. degree from the Georgia Institute of Technology, Atlanta, GA, in June 1988, and is currently enrolled in the Ph.D. program in the School of Electrical Engineering there. He has worked at Georgia Tech Research Institute of Georgia Tech, first as a co-op student and then as a Graduate Research Assistant, on a wide range of electromagnetic problems, including near-field measurements and numerical analyses.